

The Weak-Map Order and Polytopal Decompositions of Matroid Base Polytopes

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Abstract

The weak-map order on the matroid base polytopes is the partial order defined by inclusion. Lucas proved that the base polytope of no binary matroid includes the base polytope of a connected matroid. A matroid base polytope is said to be decomposable when it has a polytopal decomposition which consists of at least two matroid base polytopes. We shed light on the relation between the decomposability and the weak-map order of matroid base polytopes. We classify matroids into five types with respect to the weak-map order and decomposability. We give an example of a matroid in each class. Moreover, we give a counterexample to a conjecture proposed by Lucas, which says that, when one matroid base polytope covers another matroid base polytope with respect to inclusion, the latter matroid base polytope should be a facet of the former matroid base polytope.

1 Introduction

The set $\mathcal{B}(M)$ of the bases of a matroid M is called a *matroid base system*. The base system of M can be identified with the base polytope $B(M)$. The weak-map order that we consider is a partial ordering defined on the matroid base systems of the fixed rank and the fixed finite ground set E . The weak-map order is defined according to the inclusion relation among their bases. For matroids M_1 and M_2 , $M_1 \succeq M_2$ in the *weak-map order* is defined by $\mathcal{B}(M_2) \subseteq \mathcal{B}(M_1)$. We consider the polytopal decomposition of the base polytope of a matroid. Roughly speaking, a matroid base polytope is said to be *decomposable* when it has a polytopal decomposition which consists of at least two matroid base polytopes of the same dimension. That is, $B(M) = \bigcup_i B(M_i)$ where M_i are matroids and any intersection $B(M_i) \cap B(M_j)$ is a facet of both $B(M_i)$ and $B(M_j)$. When a matroid base polytope has a decomposition which has exactly two matroid base polytopes of the same dimension, such a polytope is said to be *2-decomposable* (or have a hyperplane-split in the literature.)

The weak-map order is related to the polytopal decomposition of a base polytope. By definition, the base polytopes obtained from a polytopal decomposition of the base polytope $B(M)$ are smaller than the base polytope $B(M)$ with respect to weak-map order. Note that the base polytope of a connected matroid is of dimension $|E| - 1$. Therefore a matroid base polytope which is minimal in all the connected matroids on E with respect to weak-map order is not decomposable.

The connected matroids are classified into the following five types. (a) Binary matroids. (b) Non-binary but minimal matroids in the connected matroids with respect to inclusion. (c) Non-minimal but indecomposable matroids. (d) Non-2-decomposable but decomposable matroids. (e) 2-decomposable matroids. An example of type (b) of rank 4 can be found in Lucas [10]. We give an example of type (b) of rank 3 in Example 50. We give an example of type (c) in Example 53. We give an example of type (d) in Example 46.

Lucas [10] conjectured that, when one matroid base polytope covers another matroid base polytope in the poset of the matroid base polytopes with respect to inclusion, the latter matroid base polytope

might be a facet of the former matroid base polytope. We give a counterexample to this conjecture in Example 56 and Theorem 59.

The problem of polytopal decomposition arises from that of M-convex functions. An *integral base polytope* is the base polytope of some integral submodular function. A matroid base polytope is the integral base polytope for the rank function of a matroid. An integral base set is the set of the lattice points of an integral base polytope. An M-convex function is a function which satisfies some kind of exchange axiom. The domain of an M-convex function is an integral base set (Murota [11, 12]). A function defined on an integral base set induces a coherent polytopal decomposition of the integral base polytope that is the convex hull of the integral base set. Such a coherent polytopal decomposition consists of integral base polytopes when the function is an M-convex function.

Kashiwabara [8] investigated a polytopal decomposition which consists of the integral base polytopes of an integral submodular function, called an integral-base decomposition, and showed that an integral base polytope can be divided into two integral base polytopes if and only if there exists a hyperplane such that the cross-section of the polytope by it is an integral base polytope. It was shown that any integral base polytope which is not a matroid base polytope by any translation is 2-decomposable. Therefore we have only to consider a polytopal decomposition of a matroid base polytope when we consider its decomposability.

This paper is organized as follows.

In Section 2, we consider the representations of the matroid independence polytopes and matroid base polytopes by linear inequalities which have 01-normal vectors. A matroid base polytope can be determined by the family of sets which satisfy linear inequalities. We call such a family a matroid base system. A flat with its rank behaves like a linear inequality for a matroid base system. We often identify a matroid base polytope with a matroid base system. We investigate the combinatorial structures of matroid base systems.

In Section 3, we consider the weak-map order and polytopal decompositions of matroid base systems. In Section 3.2, we consider the 2-decomposability of a matroid base system.

In Section 4, we consider the decomposability of a matroid base system of rank 3 in terms of graphs. Matroid base systems of rank 3 will be used as important examples in Section 5.

Section 5 is the main part of this paper. It consists of two subsections. In Section 5.1, by using Theorem 48, we classify matroid base systems into five types. In Section 5.2, we give a counterexample to the conjecture proposed by Lucas.

2 Representations of matroid systems

Let E be a finite ground set with $|E| \geq 2$ throughout this paper.

2.1 Representation of independence systems

In this subsection, we consider independence systems which may not be a matroid independence system. Moreover we introduce notation to represent a set system by linear inequalities and hyperplanes with 01-coefficients. This notation is different from standard one. However, we believe that this notation is useful to describe matroid systems from a polytopal viewpoint.

We prepare notation for hyperplanes and linear inequalities whose coefficients are in $\{0, 1\}$. A set system, which is a family of sets, can be represented by linear inequalities and hyperplanes. For a nonnegative integer a and $A \subseteq E$, we call $(A, a)_{\leq} = \{I \subseteq E \mid |I \cap A| \leq a\}$ a (closed) *linear inequality*. A family $(A, a)_{\leq}$ is identified with the 01-points $\{p \in \{0, 1\}^E \mid \langle \chi_A, p \rangle \leq a\}$ where χ_A is the incidence vector of A and $\langle \chi_A, p \rangle$ is the inner product of χ_A and p . We write $(A, a)_{>} = \{I \subseteq E \mid |I \cap A| > a\}$ and so on. Denote $(A, a)_{=} = \{I \subseteq E \mid |I \cap A| = a\}$ where a is an integer with $0 \leq a \leq |A|$ and $\emptyset \neq A \subseteq E$. A family $(A, a)_{=}$ is identified with the 01-points $\{p \in \{0, 1\}^E \mid \langle \chi_A, p \rangle = a\}$ on hyperplane $\langle \chi_A, p \rangle = a$. Therefore we call $(A, a)_{=}$ a *hyperplane*.

Lemma 1. $(A_2, a_2)_{\leq} \subseteq (A_1, a_1)_{\leq}$ if and only if $|A_1 - A_2| \leq a_1 - a_2$.

Proof. Assume $|A_1 - A_2| \leq a_1 - a_2$. Let $D \notin (A_1, a_1)_\leq$, that is, $D \in (A_1, a_1)_>$. Then $|D \cap A_1| > a_1$. Therefore $a_1 < |D \cap A_1| \leq |D \cap A_2| + |A_1 - A_2| \leq |D \cap A_2| + a_1 - a_2$. Therefore we have $|D \cap A_2| > a_2$ and $D \notin (A_2, a_2)_\leq$.

Conversely, assume $|A_1 - A_2| > a_1 - a_2$.

In the case $|A_1 - A_2| > a_1 + 1$, we can take $D \in (A_1, a_1)_>$ so that $D \subseteq A_1 - A_2$. We have $D \in (A_2, a_2)_\leq$ because of $D \cap A_2 = \emptyset$.

In the case $|A_1 - A_2| \leq a_1 + 1$, we can take $D \in (A_1, a_1)_>$ so that $|D| = a_1 + 1$ and $A_1 - A_2 \subseteq D$. At that time, we have $|D \cap A_2| + |D \cap (A_1 - A_2)| \leq |D| = a_1 + 1$ since $D \cap A_2$ and $D \cap (A_1 - A_2)$ are disjoint. By assumption $|A_1 - A_2| > a_1 - a_2$, we have $|D \cap A_2| \leq a_1 + 1 - |D \cap (A_1 - A_2)| = a_1 + 1 - |A_1 - A_2| < a_1 + 1 - (a_1 - a_2) = a_2 + 1$. Therefore $|D \cap A_2| \leq a_2$, that is, $D \in (A_2, a_2)_\leq$. \square

A non-empty family \mathcal{I} on E is called an *independence system* if $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$ imply $I_2 \in \mathcal{I}$. An independence system \mathcal{I} on E is said to be *represented* by a subset \mathcal{F} of linear inequalities if

$$I \in \mathcal{I} \Leftrightarrow I \in (A, a)_\leq \text{ for all } (A, a)_\leq \in \mathcal{F}.$$

\mathcal{F} is called a *representation* of \mathcal{I} . It is known that every independence system has such a representation. Many familiar notions, for example, rank functions, flats, and bases, in matroid theory are also defined for independence systems.

A *circuit* is a minimal dependent set. For an independence system \mathcal{I} , the *rank function* r is defined by $r(A) = \max\{|I| \mid I \in \mathcal{I}, I \subseteq A\} = \max\{|I \cap A| \mid I \in \mathcal{I}\}$. $r(E)$ is called the rank of the independence system \mathcal{I} . $A \subseteq E$ is called a *flat* for an independence system \mathcal{I} if $A = B$ holds whenever $A \subseteq B \subseteq E$ and $r(A) = r(B)$.

Every independence system \mathcal{I} is expressed as

$$\mathcal{I} = \{I \subseteq E \mid |A \cap I| \leq r(A) \text{ for all } A \subseteq E\} = \bigcap_{A \subseteq E} (A, r(A))_\leq = \bigcap_{A: \text{flat}} (A, r(A))_\leq.$$

where r is the rank function of \mathcal{I} .

For an independence system \mathcal{I} , we call $(A, a)_\leq$ *valid* for \mathcal{I} when $\mathcal{I} \subseteq (A, a)_\leq$. For a representation \mathcal{F} of \mathcal{I} , $(A, a)_\leq \in \mathcal{F}$ is valid. Note that, for an independence system \mathcal{I} , $(A, a)_\leq$ is valid if and only if $r(A) \leq a$. Especially, $(A, r(A))_\leq$ is valid.

2.2 Representations of matroid independence systems

When an independence system satisfies the matroid augmentation axiom (see, e.g. [13]), the independence system is called a *matroid independence system*.

We consider a condition for an independence system to be a matroid independence system. The next theorem is due to Conforti and Laurent [4].

A pair of linear inequalities $(A_1, a_1)_\leq$ and $(A_2, a_2)_\leq$ is said to be *intersecting* if $A_1 \cap A_2 \neq \emptyset$, $A_1 - A_2 \neq \emptyset$ and $A_2 - A_1 \neq \emptyset$.

Theorem 2. [4] *Let \mathcal{I} be an independence system represented by a set \mathcal{F} of linear inequalities. \mathcal{I} is a matroid independence system if and only if, for any intersecting inequalities $(A_1, a_1)_\leq \in \mathcal{F}$ and $(A_2, a_2)_\leq \in \mathcal{F}$, $r(A_1) + r(A_2) \geq r(A_1 \cap A_2) + r(A_1 \cup A_2)$ holds.*

It is known that, for any matroid, the intersection of any two flats is a flat again. Therefore the set of flats of a matroid is a closure system. The closure $\text{cl}(A)$ of $A \subseteq E$ is the minimum flat including A .

The minimum representation is a representation which has no redundant inequalities. Greene [7] showed the following theorem.

Theorem 3. (Greene [7]) *Every matroid independence system has the unique minimum representation defined by flats with respect to set inclusion among all the representations defined by flats. Moreover, the minimum representation is $\{(A, r(A))_\leq \mid A \text{ is the closure of a circuit}\}$.*

We consider the independence polyhedron $P(M) = \{p \in \mathbf{R}^E \mid \langle \chi_A, p \rangle \leq r(A) \text{ for all } A \subseteq E\}$ of a matroid M . We can also represent a matroid as a family of sets. By identifying a set $A \subseteq E$ with its incidence vector χ_A , the family $\mathcal{I}(M)$ is identified with $\{p \in \{0, 1\}^E \mid \langle \chi_A, p \rangle \leq r(A) \text{ for all } A \subseteq E\}$. Therefore the independent sets of a matroid can be considered as the 01-points in the polyhedron expressed by linear inequalities which have 01-normal vectors and satisfy submodularity.

For valid inequalities $\{(A_i, r(A_i))_{\leq}\}_i$ to a matroid independence system $\mathcal{I}(M)$, $\mathcal{I}(M) \cap (\bigcap_i (A_i, r(A_i))_{=})$ is called a *face* of the independence system of the matroid M . A face is *proper* if it is not $\mathcal{I}(M)$. Any face of matroid independence system $\mathcal{I}(M)$ corresponds to some face of the matroid independence polyhedron $P(M)$. A face of a matroid independence system is called a *facet* if it is maximal in all the proper faces with respect to inclusion. When $\mathcal{I}(M) \cap (A, r(A))_{=}$ is a facet, $(A, r(A))_{\leq}$ is called a *facet-defining inequality* of $\mathcal{I}(M)$.

Lemma 4. *For a loopless matroid M , if $(A, r(A))_{\leq}$ is a facet-defining inequality, A is a flat.*

Proof. Suppose that A is not a flat. $\text{cl}(A)$ denotes the closure of A . Then $r(\text{cl}(A)) = r(A)$. We can take $x \in \text{cl}(A) - A$ since A is not a flat. Since the restriction of loopless matroid M to $\text{cl}(A)$ is also a loopless matroid, there exists $B \in \mathcal{I}(M) \cap (\text{cl}(A), r(A))_{=}$ with $x \in B \subseteq \text{cl}(A)$. Then $B \notin \mathcal{I}(M) \cap (A, r(A))_{=}$ since $|B \cap A| < |B \cap (A \cup x)| \leq |B \cap \text{cl}(A)| = |B| = r(A)$. Since, for any $X \in \mathcal{I}(M)$, $|X \cap A| = r(A)$ implies $|X \cap \text{cl}(A)| = r(A)$, we have $\mathcal{I}(M) \cap (A, r(A))_{=} \subseteq \mathcal{I}(M) \cap (\text{cl}(A), r(\text{cl}(A)))_{=}$. Hence we have $\mathcal{I}(M) \cap (A, r(A))_{=} \subsetneq \mathcal{I}(M) \cap (\text{cl}(A), r(\text{cl}(A)))_{=}$. Therefore $(A, r(A))_{\leq}$ is not a facet-defining inequality. \square

For a loopless matroid, we call a flat A a *facet-defining flat* of $\mathcal{I}(M)$ if $(A, r(A))_{\leq}$ is a facet-defining inequality of $\mathcal{I}(M)$.

The next theorem is due to Edmonds.

Theorem 5. (Edmonds [6]) *For a loopless matroid independence system $\mathcal{I}(M)$, $\mathcal{I}(M) \cap (A, r(A))_{=}$ is a facet of $\mathcal{I}(M)$ if and only if A is a flat and the restriction $M|A$ of M to A is connected.*

We discuss the relation between Theorem 3 and Theorem 5. When A is the closure of some circuit, A is a flat and $M|A$ is connected since any circuit is included in some connected component. Therefore we have $\mathcal{I}(M) = \bigcap_{\{A \text{ is a facet-defining flat}\}} (A, r(A))_{\leq}$. However, it may not be the minimum representation by flats.

Example 6. *Consider the matroid M on $E = \{a, b, c, d, e, f\}$ defined by*

$$\mathcal{I}(M) = (E, 4)_{\leq} \cap (\{a, b, c, d\}, 3)_{\leq} \cap (\{c, d, e, f\}, 3)_{\leq} \cap (\{e, f, a, b\}, 3)_{\leq}.$$

E is the closure of no circuit. However, E is a flat and $M|E$ is a connected matroid. Therefore $(E, 4)_{\leq}$ is a facet-defining inequality of $\mathcal{I}(M)$ but does not belong to the minimum representation by flats. In fact, $\mathcal{I}(M) = (\{a, b, c, d\}, 3)_{\leq} \cap (\{c, d, e, f\}, 3)_{\leq} \cap (\{e, f, a, b\}, 3)_{\leq}$.

2.3 Representations of matroid base systems

For a matroid, a *base* is a maximal independent set with respect to inclusion. For a matroid M , the family of its bases is denoted by $\mathcal{B}(M)$ and called the *matroid base system* of M . We call a family of sets a *matroid base system* when it is the matroid base system of some matroid. Since any base of a matroid has the same cardinality $r(E)$, we have $\mathcal{B}(M) = \mathcal{I}(M) \cap (E, r(E))_{=}$.

We can identify the base system $\mathcal{B}(M)$ of a matroid M with the base polytope

$$B(M) = \{p \in \mathbf{R}^E \mid \langle p, \chi_A \rangle \leq r(A) \text{ for all } A \subseteq E, \langle p, \chi_E \rangle = r(E)\}$$

since the bases of a matroid correspond to the extreme points of the base polytope so that $B(M) = \text{conv}\{\chi_B \mid B \in \mathcal{B}(M)\}$.

We try to describe the combinatorial structures of a matroid base system in terms of a family of sets. For a set of valid inequalities $\{(A_i, r(A_i))_{\leq}\}_i$ to a matroid M , $\mathcal{B}(M) \cap (\bigcap_i (A_i, a_i)_{=})$ is called a *face* of the matroid base system $\mathcal{B}(M)$. A face is said to be *proper* when it is not equal to $\mathcal{B}(M)$.

Proposition 7. *The faces of the form $\mathcal{B}(M) \cap (\bigcap_i (A_i, r(A_i))_=)$ correspond to the faces of matroid base polytope $B(M)$ bijectively.*

Proof. Assume that a face of $B(M)$ is given. Since every facet of $B(M)$ has a normal vector χ_A with 01-coefficients, it can be written as $\mathcal{B}(M) \cap (A, r(A))_=$. Since every face of $B(M)$ can be written as the intersection of some facets of $B(M)$, the face corresponds to some $\mathcal{B}(M) \cap (\bigcap_i (A_i, r(A_i))_=)$.

Conversely, assume that a face of the form $\mathcal{B}(M) \cap (\bigcap_i (A_i, r(A_i))_=)$ is given. By considering the convex hull of the face, we have the corresponding face of $B(M)$. \square

It is known that every face of a matroid base system is also a matroid base system.

A face of a matroid base system is called a *facet* if it is maximal in all the proper faces with respect to set inclusion. A facet is expressed as $\mathcal{B}(M) \cap (A, r(A))_=$. We call an inequality $(A, r(A))_\leq$ a *facet-defining inequality* when $\mathcal{B}(M) \cap (A, r(A))_=$ is a facet. For a matroid, A is a flat when $(A, r(A))_\leq$ is a facet-defining inequality by Lemma 4 and Theorem 13. The facets of a matroid base system are in one-to-one correspondence with the facets of the base polytope $B(M)$.

A matroid is not *connected* if it is expressed as $\mathcal{B}(M) = \mathcal{B}(M_1) \oplus \mathcal{B}(M_2) = \{B_1 \cup B_2 | B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}$ such that the non-empty underlying sets of M_1 and M_2 are disjoint. The number of the connected components of M is closely related to the dimension of the base polytope (e.g. [2]):

$$(\text{the dimension of the base polytope}) = |E| - (\text{the number of its connected components}).$$

A face of a base polytope is a facet when the dimension of the face is less than that of the base polytope by 1. Therefore we know that a face of a matroid base polytope is a facet if and only if the number of the connected components of the matroid corresponding to the face is more than that of connected components of the whole matroid by 1. Especially, the number of its connected components of the matroid corresponding to a facet of the base polytope of a connected matroid is 2. We define the dimension of a base system as the dimension of the corresponding base polytope, that is equal to $(|E| - \text{the number of connected components})$. Note that the dimension of a base system is irrelevant to the rank of the matroid.

For a connected matroid M , a facet-defining inequality $(A, r(A))_\leq$ of $\mathcal{B}(M)$ is *non-trivial* if $(E, r(E))_= \cap (A, r(A))_>$ and $(E, r(E))_= \cap (A, r(A))_<$ are not empty. Otherwise a facet-defining inequality is said to be *trivial*. The next lemma follows from the definition.

Lemma 8. $(E, r(E))_= \cap (A, r(A))_\leq = (E, r(E))_= \cap (A^c, r(E) - r(A))_\geq$.

Example 9. Let $E = \{a, b, c, d, e\}$. Consider the independence system $\mathcal{I}(M) = (E, 3)_\leq \cap (\{a, b, c\}, 2)_\leq \cap (\{c, d, e\}, 2)_\leq$, illustrated in the left of Figure 1. A flat $(A, r(A))_\leq$ in the minimal representation of the independence system is represented by a closed curve that encircles the elements of A in the figure. By Theorem 2, it is a matroid independence system. We want to specify all the facets of matroid base system

$$\mathcal{B}(M) = \mathcal{I}(M) \cap (E, 3)_= = (E, 3)_= \cap (\{a, b, c\}, 2)_\leq \cap (\{c, d, e\}, 2)_\leq.$$

This matroid has two non-trivial facets, $\mathcal{B}(M) \cap (\{a, b, c\}, 2)_=$ and $\mathcal{B}(M) \cap (\{c, d, e\}, 2)_=$. Let M_1 be the matroid which corresponds to a facet $\mathcal{B}(M) \cap (\{a, b, c\}, 2)_= = \mathcal{B}(M_1)$ of $\mathcal{B}(M)$. Since $(E, 3)_= \cap (\{a, b, c\}, 2)_\geq = (E, 3)_= \cap (\{d, e\}, 1)_\leq$ by Lemma 8, $\mathcal{B}(M_1) = (E, 3)_= \cap (\{a, b, c\}, 2)_\leq \cap (\{d, e\}, 1)_\leq = (\{a, b, c\}, 2)_= \cap (\{d, e\}, 1)_=$, illustrated in the center of Figure 1. This is a non-connected matroid, which has two connected components $\{a, b, c\}$ and $\{d, e\}$. Therefore $\mathcal{B}(M_1)$ is really a facet of $\mathcal{B}(M)$.

We consider the face $\mathcal{B}(M_2)$ defined by $\mathcal{B}(M) \cap (\{e\}, 1)_=$. Since $(\{c, d\}, 1)_\leq \subseteq (\{c, d, e\}, 2)_\leq$,

$$\begin{aligned} \mathcal{B}(M_2) &= \mathcal{B}(M) \cap (\{e\}, 1)_= = (E, 3)_= \cap (\{a, b, c\}, 2)_\leq \cap (\{c, d, e\}, 2)_\leq \cap (\{e\}, 1)_= \\ &= (E, 3)_= \cap (\{a, b, c, d\}, 2)_= \cap (\{c, d, e\}, 2)_\leq \cap (\{e\}, 1)_= \\ &= (E, 3)_= \cap (\{a, b, c, d\}, 2)_= \cap (\{c, d, e\}, 2)_\leq \cap (\{c, d\}, 1)_\leq \cap (\{e\}, 1)_= \\ &= (\{a, b, c, d\}, 2)_= \cap (\{c, d\}, 1)_\leq \cap (\{e\}, 1)_= \end{aligned}$$

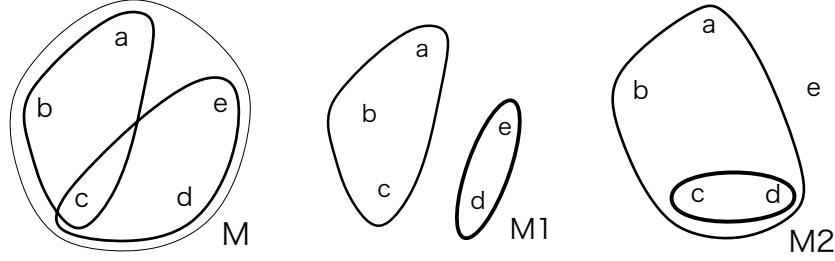


Figure 1: $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ are facets of a matroid base system $\mathcal{B}(M)$.

Since $\mathcal{B}(M_2)$ has two connected components $\{a, b, c, d\}$ and $\{e\}$, $\mathcal{B}(M_2) = (\{e\}, 1)_= \cap \mathcal{B}(M)$ is a trivial facet of $\mathcal{B}(M)$.

However, $(\{c\}, 1)_= \cap \mathcal{B}(M_1) = (\{a, b\}, 1)_= \cap (\{c\}, 1)_= \cap (\{d, e\}, 1)_=$ is not a facet of $\mathcal{B}(M)$ since it has three connected components, $\{a, b\}$, $\{c\}$ and $\{d, e\}$ as a matroid.

$(\{a, b, d, e\}, 3)_= \cap \mathcal{B}(M) = (E, 3)_= \cap (\{a, b, d, e\}, 3)_\leq \cap (\{c\}, 0)_\leq = (\{a, b, d, e\}, 3)_= \cap (\{c\}, 0)_=$. Therefore $(\{a, b, d, e\}, 3)_\leq$ is a trivial facet-defining inequality of $\mathcal{B}(M)$.

In summary, the base polytope $B(M)$ of M has the following facets.

$$B(M) = \left\{ p \in \mathbf{R}^E \mid \begin{array}{l} p(\{a, b, d, e\}) \leq 3 \\ p(\{a\}) \leq 1, p(\{b\}) \leq 1, p(\{d\}) \leq 1, p(\{e\}) \leq 1 \\ p(\{a, b, c\}) \leq 2, p(\{c, d, e\}) \leq 2, p(E) = 3 \end{array} \right\}$$

where $p(A) = \langle p, \chi_A \rangle$.

Proposition 10. For a matroid M , $\mathcal{I}(M) \subseteq (A, a)_\leq$ if and only if $\mathcal{B}(M) \subseteq (A, a)_\leq$.

Proof. Only-If-part follows from $\mathcal{B}(M) \subseteq \mathcal{I}(M)$.

We show If-part. Assume that $\mathcal{I}(M) \subseteq (A, a)_\leq$ does not hold. Then $|I \cap A| > a$ for some $I \in \mathcal{I}(M)$. Then there exists $B \in \mathcal{B}(M)$ such that $I \subseteq B$. We have $|B \cap A| > a$. Therefore $B \in \mathcal{B}(M)$ and $B \notin (A, a)_\leq$. \square

Next, we consider the relation between base systems and minor operations on matroids.

Consider contraction M/A and deletion $M \setminus A$ by $A \subseteq E$ for a matroid M . $M|A$ denotes the restriction of M by A , which is equal to $M \setminus A^c$. Define $\mathcal{B}(M) \setminus A := \mathcal{B}(M \setminus A)$, and so on.

The next lemma follows from the definition of contraction.

Lemma 11. For a matroid M and $A \subseteq E$,

$$\mathcal{B}(M) \cap (A, r(A))_= = \mathcal{B}(M|A) \oplus \mathcal{B}(M/A).$$

Proof. Note that $\mathcal{B}(M|A) = (\mathcal{B}(M) \cap (A, r(A))_=)|A$ and $\mathcal{B}(M/A) = (\mathcal{B}(M) \cap (A, r(A))_=)|A^c$. $\mathcal{B}(M) \cap (A, r(A))_=$ is a matroid base system since every face of a matroid base system is a matroid base system. Let M' be the matroid $\mathcal{B}(M) \cap (A, r(A))_=$ with the rank function r' . Since $\mathcal{B}(M') \subseteq (E, r(E)) \cap (A, r(A))_=$, we have $r'(A) + r'(A^c) = r'(E)$ by Lemma 8. We have $\mathcal{B}(M) \cap (A, r(A))_= = \mathcal{B}(M|A) \oplus \mathcal{B}(M/A)$ since $r'(A) + r'(A^c) = r'(E)$. \square

The base system $\mathcal{B}(M^*)$ of the dual matroid M^* is given by $\{B^c | B \in \mathcal{B}(M)\}$. It is known that the connectivity of $\mathcal{B}(M^*)$ is equivalent to that of $\mathcal{B}(M)$.

Lemma 12. For a matroid M and $A \subseteq E$, $M^*|A^c$ is connected if and only if M/A is connected.

Proof. $M^*|A^c = M^* \setminus A = (M/A)^*$. Note that the connectivity of $(M/A)^*$ is equal to that of M/A . \square

Theorem 13. *For a connected matroid M on E , the following are equivalent.*

- (a) $(A, r(A))_{\leq}$ is a facet-defining inequality of the base system $\mathcal{B}(M)$.
- (b) $\mathcal{B}(M) \cap (A, r(A))_{=}$ has the two connected components A and A^c .
- (c) $M|A$ and M/A are both connected.
- (d) $(A, r(A))_{\leq}$ is a facet-defining inequality of the independence system $\mathcal{I}(M)$ and $(A^c, r^*(A^c))_{\leq}$ is a facet-defining inequality of the independence system $\mathcal{I}(M^*)$ of the dual matroid M^* .

Proof. [(a) \leftrightarrow (b)] Since $(A, r(A))_{\leq}$ is a valid inequality to $\mathcal{B}(M)$, $\mathcal{B}(M) \cap (A, r(A))_{=}$ is a proper face of $\mathcal{B}(M)$. Any facet of the base system of the connected matroid is a matroid base system of dimension $|E| - 2$. By the relation between the dimension of a matroid base system and the number of its connected components, $(A, r(A))_{\leq}$ is a facet-defining inequality of the base system if and only if $\mathcal{B}(M) \cap (A, r(A))_{=}$ has two connected components.

[(b) \leftrightarrow (c)] By Lemma 11, (b) is equivalent to (c).

[(c) \leftrightarrow (d)] Note that A is a flat of M when M/A is connected. Moreover A^c is a flat of M when $M|A$ is connected. $(A, r(A))_{\leq}$ is a facet-defining inequality of the independence system of the matroid M if and only if A is a flat and $M|A$ is connected by Theorem 5. $(A^c, r^*(A^c))_{\leq}$ is a facet-defining inequality of the independence system of the dual matroid M^* if and only if A^c is a flat and $M^*|A^c$ is connected by Theorem 5. By Lemma 12, $M^*|A^c$ is connected if and only if M/A is connected. \square

Consequently, when $(A, r(A))_{\leq}$ is a facet-defining inequality of $\mathcal{B}(M)$, A is a flat. We call such a flat a *facet-defining flat* of $\mathcal{B}(M)$. Note that $|A| > r(A)$ for any facet-defining inequality $(A, r(A))_{\leq}$.

Note that the decomposition in terms of matroid base polytopes is totally different from the decomposition by connected components.

We give an example of a facet of a matroid independence system that is not a facet of the matroid base system.

Example 14. Let $E = \{a, b, c, d, e, f\}$. Consider the matroid base system $\mathcal{B}(M) = (E, 3)_{=} \cap (\{a, b, c, d\}, 2)_{\leq} \cap (\{a, b, e, f\}, 2)_{\leq}$ (the left of Figure 2). However, the independence system $(E, 3)_{\leq} \cap (\{a, b, c, d\}, 2)_{\leq} \cap (\{a, b, e, f\}, 2)_{\leq}$ does not satisfy submodularity

$$r'(\{a, b, c, d\}) + r'(\{a, b, e, f\}) \geq r'(\{a, b, c, d, e, f\}) + r'(\{a, b\})$$

as in Theorem 2 since $r'(\{a, b\}) = 2$. Therefore it is not a matroid independence system. The matroid independence system corresponding to $\mathcal{B}(M)$ has the rank function $r(A) = \max\{|B \cap A| \mid B \in \mathcal{B}(M)\}$. Therefore $r(\{a, b\}) = 1$. We have

$$\mathcal{I}(M) = (\{a, b, c, d\}, 2)_{\leq} \cap (\{a, b, e, f\}, 2)_{\leq} \cap (\{a, b\}, 1)_{\leq}, \text{ and}$$

$$\mathcal{B}(M) = \mathcal{I}(M) \cap (E, 3)_{=} = (E, 3)_{=} \cap (\{a, b, c, d\}, 2)_{\leq} \cap (\{a, b, e, f\}, 2)_{\leq} \cap (\{a, b\}, 1)_{\leq}.$$

$\mathcal{I}(M) \cap (\{a, b\}, 1)_{=}$ is a facet of the independence system $\mathcal{I}(M)$ of matroid M . Note that $\{a, b\}$ is a flat and $M|_{\{a, b\}}$ is connected but not a facet-defining inequality of $\mathcal{B}(M)$ since $M/\{a, b\}$ is not connected. In fact, $\mathcal{B}(M) \cap (\{a, b\}, 1)_{=}$ has three connected components. Therefore $\mathcal{B}(M) \cap (\{a, b\}, 1)_{=}$ is not a facet of $\mathcal{B}(M)$. The right of Figure 2 depicts the dual matroid M^* , satisfying $\mathcal{B}(M^*) = (E, 3)_{=} \cap (\{c, d\}, 1)_{\leq} \cap (\{e, f\}, 1)_{\leq}$. Note that $\mathcal{B}(M^*)|_{\{c, d, e, f\}} = (\{c, d\}, 1)_{=} \cap (\{e, f\}, 1)_{=}$ is not connected.

3 The weak-map order and polytopal decompositions of matroid base systems

3.1 The weak-map order and polytopal decompositions of matroid base systems of general rank

Consider a partial ordering, called the *weak-map order*, on the matroid base systems of rank r on E (See [5]). For matroids M_1 and M_2 of the same rank on E , $M_1 \succeq M_2$ in the weak-map order is defined by

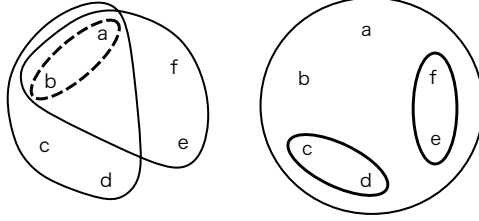


Figure 2: $\mathcal{B}(M)$ and $\mathcal{B}(M^*)$. $\mathcal{B}(M) \cap (\{a, b\}, 1)_=$ is a facet of the independence system $\mathcal{I}(M)$ but not a facet of the base system $\mathcal{B}(M)$.

$\mathcal{B}(M_2) \subseteq \mathcal{B}(M_1)$. The maximum element of the weak-map order on all the matroids of rank r on E is the uniform matroid of rank r , denoted by $U_{r,|E|}$.

The proof of the next lemma is straightforward.

Lemma 15. *Assume that a matroid M_1 on E has a rank function r_1 and a matroid M_2 on E has a rank function r_2 with $r_1(E) = r_2(E)$. $\mathcal{B}(M_2) \subseteq \mathcal{B}(M_1)$ if and only if $r_2(A) \leq r_1(A)$ for all $A \subseteq E$.*

Next, we consider a polytopal decomposition of a matroid base system.

As already defined, a matroid base polytope is said to be decomposable if it has a polytopal decomposition which consists of matroid base polytopes. Next we rewrite a decomposition in terms of matroid base systems.

Definition 16. *A matroid base system $\mathcal{B}(M)$ is decomposable if there exist matroid base systems $\{\mathcal{B}_i\}_i$ satisfying the following conditions.*

- (a). $\mathcal{B}(M) = \bigcup_i \mathcal{B}_i$,
- (b). For any i , the connected components of matroid M_i corresponding to \mathcal{B}_i are the same as those of M .
- (c). For any distinct i and j , $\mathcal{B}_i \cap \mathcal{B}_j$ is a proper face of both \mathcal{B}_i and \mathcal{B}_j , and there exists a hyperplane $(A, a)_=$ such that $\mathcal{B}_i \subseteq (A, a)_\leq$ and $\mathcal{B}_j \subseteq (A, a)_\geq$.
- (d). For any i and each facet \mathcal{B} of \mathcal{B}_i with a facet-defining inequality $(A, a)_\leq$, \mathcal{B} is a facet of $\mathcal{B}(M)$ or a facet of \mathcal{B}_j for some unique j with the facet-defining inequality $(A, a)_\geq$.

Proposition 17. *A matroid base system $\mathcal{B}(M)$ is decomposed into $\{\mathcal{B}(M_i)\}$ consisting of matroid base systems if and only if $B(M)$ has a polytopal decomposition $\{B(M_i)\}$ where $B(M) = \text{conv}(\mathcal{B}(M))$ and $B(M_i) = \text{conv}(\mathcal{B}(M_i))$ for any i .*

Proof. Assume that $\mathcal{B}(M)$ is decomposed into $\{\mathcal{B}(M_i)\}$ so that Conditions (a) to (d) in Definition 16 are satisfied. Let $B(M) = \text{conv}(\mathcal{B}(M))$ and $B(M_i) = \text{conv}(\mathcal{B}(M_i))$. We first show $B(M) = \bigcup B(M_i)$. Every extreme point of $B(M)$ belongs to $\bigcup B(M_i)$ by Condition (a). Therefore, it suffices to show that $\bigcup B(M_i)$ is convex. By Condition (b), $B(M_i)$ and $B(M)$ have the same dimension for any i . Even though $\bigcup \{\mathcal{B}(M_i)\}$ may not be convex, we can consider the inequalities which define the boundary of $\bigcup \{\mathcal{B}(M_i)\}$ on each connected component. Any inequality that defines the boundary of $\bigcup \{\mathcal{B}(M_i)\}$ is a facet-defining inequality of $B(M)$ by Condition (d). Hence $\bigcup B(M_i)$ is convex. Therefore, we have $B(M) = \bigcup B(M_i)$.

By Condition (d), $B(M_i) \cap B(M_j) = \text{conv}(\mathcal{B}_i \cap \mathcal{B}_j)$ for distinct i and j . By Condition (c), $B(M_i) \cap B(M_j)$ is a face of $B(M_i)$ and $B(M_j)$, and $B(M_i) \cap B(M_j) = \text{conv}(\mathcal{B}_i \cap \mathcal{B}_j)$. Therefore $B(M)$ has a polytopal decomposition $\{B(M_i)\}$.

Conversely, we assume that $B(M)$ has a polytopal decomposition $\{B(M_i)\}$. Let $\mathcal{B}(M)$ be the collection of (the supports of) the extreme points of $B(M)$, and $\mathcal{B}(M_i)$ be the collection of the extreme points of $B(M_i)$ for each i . Then, Conditions (a) to (d) in Definition 16 are obviously satisfied. \square

The next lemma is trivial but important when we consider the relation between the weak-map order and polytopal decompositions.

Lemma 18. *Every matroid base polytope belonging to a polytopal decomposition of a matroid base polytope $B(M)$ is smaller than the matroid base polytope $B(M)$ with respect to weak-map order.*

Lemma 19. *A loopless matroid base system $\mathcal{B}(M)$ is decomposable if and only if the base system of the simplified matroid is decomposable.*

Proof. Assume that a matroid base system $\mathcal{B}(M)$ is decomposed into $\{\mathcal{B}(M_i)\}$. It suffices to show that $x, y \in E$ are parallel on M_i when $x, y \in E$ are parallel on M for M_i in the decomposition with its rank function r_i . Since M_i has no loop, $0 < r_i(\{x, y\}) \leq r(\{x, y\})$. Therefore $r_i(\{x, y\}) = 1$.

The converse direction is obvious. \square

Therefore we have only to consider simple matroids.

Next we consider the relation between decomposability and matroid connectivity.

Lemma 20. *A matroid base system $\mathcal{B}(M)$ is not decomposable if and only if the matroid base system of no connected component of the matroid M is decomposable.*

Proof. When the base system of some connected component of M is decomposable, we can make a decomposition of $\mathcal{B}(M)$ in terms of the decomposition of such a connected component.

Conversely, assume that a matroid base system $\mathcal{B}(M)$ with its rank function r is decomposed into $\mathcal{D} = \{\mathcal{B}(M_i)\}$. Let $\{E_k\}$ be the partition consisting of the separators induced by the connected components, which are the underlying sets of the connected components of M . We can take k and l so that $\mathcal{B}(M|E_k) \neq \mathcal{B}(M_l|E_k)$. $(A, r_l(A))_{=}$ is a facet-defining equality of M_l with its rank function r_l and $\mathcal{B}(M_j) \subseteq (A, r_l(A))_{\geq}$ for some $M_j \in \mathcal{D}$. Therefore $(A, r_l(A))_{=}$ is also a facet-defining equality of some $M_l|E_k$. Therefore $\{\mathcal{B}(M_i|E_k) \mid \mathcal{B}(M_l|E_k) = \mathcal{B}(M_i|E_k^c), \mathcal{B}(M_i) \in \mathcal{D}\}$ is a decomposition of $\mathcal{B}(M|E_k)$. \square

Therefore we have only to consider connected matroids when we consider decomposability. Chatlain and Alfonsin[3] investigated the relation between decomposability of a matroid base polytope and its connected components.

3.2 2-decomposability of matroid base systems

As already defined, when a matroid base polytope has a polytopal decomposition which consists of exactly two matroid base polytopes, such a polytope is said to be *2-decomposable*.

We can rewrite this in terms of matroid base systems as follows.

Definition 21. *For a matroid M , $\mathcal{B}(M)$ is 2-decomposable if there exists a hyperplane $(A, a)_{=}$ such that*

- (a). $\mathcal{B}(M) \cap (A, a)_{\geq}$ is a matroid base system,
- (b). $\mathcal{B}(M) \cap (A, a)_{\leq}$ is a matroid base system,
- (c). $\mathcal{B}(M) \cap (A, a)_{>}$ is not empty, and
- (d). $\mathcal{B}(M) \cap (A, a)_{<}$ is not empty.

Theorem 22. [8] *A matroid base system $\mathcal{B}(M)$ can be decomposed into two matroid base systems if and only if there exists a hyperplane $(A, r)_{=}$ such that*

- $\mathcal{B}(M) \cap (A, r)_{=}$ is a matroid base system,
- $\mathcal{B}(M) \cap (A, r)_{>}$ and $\mathcal{B}(M) \cap (A, r)_{<}$ are non-empty.

Kim[9] also give a necessary and sufficient condition for 2-decomposability. We give an example of a 2-decomposable matroid.

Example 23. Let $E = \{a, b, c, d, e\}$. Consider the matroid independence system $\mathcal{I}(M) = (E, 3)_{\leq} \cap (\{a, b, c\}, 2)_{\leq}$, illustrated in the first figure of Figure 3. Then $\mathcal{B}(M) = \mathcal{I}(M) \cap (E, 3)_{=} = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq}$. Even by adding $(\{c, d, e\}, 2)_{=}$ into the representation, the independence system $\mathcal{I}(M) \cap (\{c, d, e\}, 2)_{\leq} = (E, 3)_{\leq} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e\}, 2)_{\leq}$, illustrated in the second figure, still satisfies submodularity as in Theorem 2. By Theorem 22, the base system $\mathcal{B}(M)$ of this matroid is divided into two base systems $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ by hyperplane $(\{c, d, e\}, 2)_{=}$. The base system of one matroid M_1 is

$$\mathcal{B}(M_1) = \mathcal{B}(M) \cap (\{c, d, e\}, 2)_{\leq} = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e\}, 2)_{\leq},$$

illustrated in the second figure of Figure 3. The base system of the other matroid M_2 is

$$\begin{aligned} \mathcal{B}(M_2) &= \mathcal{B}(M) \cap (\{c, d, e\}, 2)_{\geq} = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e\}, 2)_{\geq} \\ &= (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{a, b\}, 1)_{\leq} = (E, 3)_{=} \cap (\{a, b\}, 1)_{\leq}, \end{aligned}$$

illustrated in the third figure of Figure 3, since $(\{a, b\}, 1)_{\leq} \subseteq (\{a, b, c\}, 2)_{\leq}$ by Lemma 1. The matroid base system corresponding to the cross-section by the hyperplane $(\{c, d, e\}, 2)_{=}$ is

$$\begin{aligned} \mathcal{B}(M_1 \cap M_2) &= \mathcal{B}(M) \cap (\{c, d, e\}, 2)_{=} = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e\}, 2)_{=} \\ &= (\{a, b\}, 1)_{\leq} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e\}, 2)_{=} = (\{a, b\}, 1)_{\leq} \cap (\{c, d, e\}, 2)_{=} \end{aligned}$$

since $(\{a, b\}, 1)_{\leq} \cap (\{c, d, e\}, 2)_{\leq} \subseteq (E, 3)_{\leq}$. $\mathcal{B}(M_1 \cap M_2)$ is illustrated in the last figure of Figure 3.

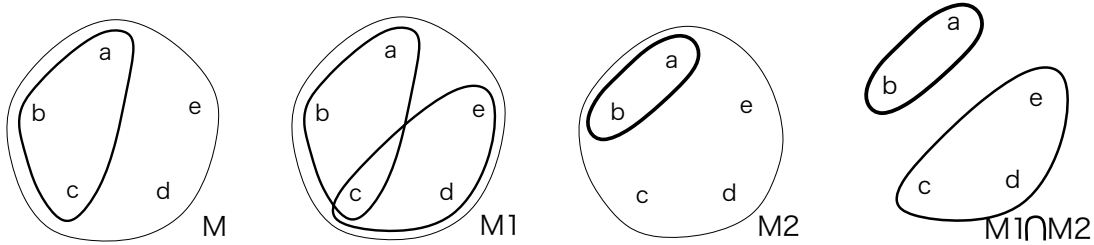


Figure 3: Dividing a matroid base system $\mathcal{B}(M)$ into two matroid base systems $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ with intersection $\mathcal{B}(M_1) \cap \mathcal{B}(M_2)$.

4 Matroid base systems of rank 3

In this section, we focus on matroid base systems of rank 3 only. Matroid base systems of rank 3 will be used as important examples in Section 5.

4.1 Representation of matroid base systems of rank 3

Lemma 24. For a matroid of rank 3, for any flat F_1 of rank 2 and any set F_2 of rank 2 with $F_2 - F_1 \neq \emptyset$, $F_1 \cap F_2$ is of rank at most 1. Especially, for a matroid of rank 3, for any distinct flats F_1, F_2 of rank 2, $F_1 \cap F_2$ is of rank at most 1. Moreover, M is simple, $|F_1 \cap F_2| \leq 1$.

Proof. Since F_1 is a flat and $F_2 - F_1 \neq \emptyset$, $F_1 \cup F_2$ has rank 3. Therefore, $F_1 \cap F_2$ is of rank at most 1 by submodularity $r(F_1) + r(F_2) \geq r(F_1 \cup F_2) + r(F_1 \cap F_2)$. \square

Lemma 25. Consider a connected matroid M of rank 3. Let A be a flat of rank 1. Then $(A, 1)_{\leq}$ is not a facet-defining inequality of $\mathcal{B}(M)$ if and only if $\mathcal{B}(M) \cap (A^c, 2)_{\leq}$ has exactly two connected components.

Proof. We have $\mathcal{B}(M) \cap (A^c, 2)_\leq = \mathcal{B}(M) \cap (A, 1)_\geq = \mathcal{B}(M) \cap (A, 1)_=$ since the rank of A is 1. Therefore the statement follows from Theorem 13. \square

Corollary 26. *Consider a connected matroid M of rank 3 on E . Let $A \subseteq E$ be a flat of rank 1. Then $(A, 1)_\leq$ is a facet-defining inequality of $\mathcal{B}(M)$ if and only if there exists no pair of flats F_1, F_2 of rank 2 such that $A = F_1 \cap F_2$ and $F_1 \cup F_2 = E$.*

Proof. Assume that $A = F_1 \cap F_2$, and $F_1 \cup F_2 = E$. Let r be the rank function of M . It suffices to show that the matroid $\mathcal{B}(M) \cap (A, 1)_=$ has three connected components by Lemma 25. Let r' be the rank function of $\mathcal{B}(M) \cap (A, 1)_=$. Note that $\mathcal{B}(M) \cap (A, 1)_= = \mathcal{B}(M) \cap (A^c, 2)_=$ by Lemma 8. Therefore, on $\mathcal{B}(M) \cap (A^c, 2)_=$, $A^c \cap F_1$ has rank 1 because of the submodularity $r'(F_1) + r'(A^c) \geq r'(F_1 \cap A^c) + r'(E)$. Similarly, $A^c \cap F_2$ has rank 1 on $\mathcal{B}(M) \cap (A^c, 2)_=$. Since $r'(A^c \cap F_1) + r'(A^c \cap F_2) = r'(A^c)$, $\mathcal{B}(M) \cap (A^c, 2)_\leq$ has three connected components $A, A^c \cap F_1$ and $A^c \cap F_2$.

Conversely, assume that $(A, 1)_\leq$ is not a facet-defining inequality. Then $\mathcal{B}(M) \cap (A^c, 2)_\leq$ has exactly two connected components of rank 1 on A^c by Lemma 25. Let A_1 and A_2 be such connected components. Note that A is also a flat of rank 1 on $\mathcal{B}(M) \cap (A^c, 2)_\leq$.

We first show that $F_1 := A \cup A_1$ and $F_2 := A \cup A_2$ have both rank 2 on M . Note that $r(A_1) = r'(A_1) = 1$ and $r(A_2) = r'(A_2) = 1$ since $A_1, A_2 \subseteq A^c$. $F_2 = A \cup A_2$ has rank 2 on M because A is a flat of rank 1 and submodularity $r(F_2) \leq r(A) + r(A_2)$. $F_1 = A \cup A_1$ has rank 2 on M because of submodularity $r(A) + r(A_1) \geq r(F_1)$. Note that $r(F_1) > r(A) = 1$ since A is a flat on M . The only remaining part to show is that F_1 and F_2 are flats on M . On the contrary, suppose that there exists $F \subseteq E$ such that $A \cup A_1 \subsetneq F$ and $r(F) = 2$. Then $F \cap (A_1 \cup A_2)$ has rank 1 by submodularity, which contradicts that A_1 is a flat of rank 1 on M . \square

Proposition 27. *For a connected simple matroid, every flat F of rank 2 is facet-defining when $(F, 2)_\leq$ is non-trivial, i.e. $|F| \geq 3$.*

Proof. We first show the restriction $M|F$ of M to F is connected. On the contrary, suppose that $M|F$ has two connected components A_1 and A_2 of rank 1 on M . Since $|F| \geq 3$, $|A_1| \geq 2$ or $|A_2| \geq 2$, which contradicts that M is simple.

Therefore, $\mathcal{B}(M) \cap (F, 2)_=$ has exactly two connected components. Hence $(F, 2)_\leq$ is a facet-defining inequality by Theorem 13. \square

Lemma 28. *Consider a connected matroid M of rank 3 on E with a rank function r . For a facet-defining inequality $(F_2, 2)_\leq$ and $F_1 \subseteq E$ such that $r(F_1) = 1$ and $F_1 \cap F_2 \neq \emptyset$, we have $F_1 \subseteq F_2$ and $|F_2 - F_1| \geq 2$.*

Proof. When F_2 and F_1 do not satisfy $F_1 \subseteq F_2$, we have $r(F_1 \cup F_2) = 3$ since F_2 is a flat and the rank of M is 3. Since $F_1 \cap F_2 \neq \emptyset$ and M is connected, we have $r(F_1 \cap F_2) = 1$. Therefore it contradicts submodularity $r(F_1) + r(F_2) \geq r(F_1 \cap F_2) + r(F_1 \cup F_2)$. Hence $F_1 \subseteq F_2$.

When $|F_2 - F_1| = 1$ and $F_1 \subseteq F_2$, matroid base system $\mathcal{B}(M) \cap (F_2, 2)_=$ has a connected component $F_2 - F_1$. Since $(F_1, 1)_\leq \subseteq (F_2, 2)_\leq$ by Lemma 1, $(F_2, 2)_\leq$ is not a facet-defining inequality. \square

4.2 2-decomposability of matroids of rank 3

In this subsection, we consider the 2-decomposability of a matroid base system of rank 3.

Note that a connected matroid base system of rank 3 is 2-decomposable by $(A, 1)_=$ if and only if it is 2-decomposable by $(A^c, 2)_=$ since $(E, 3)_= \cap (A, 1)_= = (E, 3)_= \cap (A^c, 2)_=$ by Lemma 8.

Theorem 29. *A connected simple matroid base system $\mathcal{B}(M)$ of rank 3 is 2-decomposable by $(A, 2)_=$ if and only if it satisfies the following conditions.*

- (1) $|A^c| \geq 2$,
- (2) the rank of A is 3,
- (3) for any facet-defining inequality $(F, 2)_\leq$ of $\mathcal{B}(M)$, one of the following is satisfied. (3-1) $|A \cap F| \leq 1$, (3-2) $F \subseteq A$, (3-3) $A \cup F = E$.

Proof. Assume that (1),(2), and (3) are satisfied. We show that $\mathcal{B}(M) \cap (A, 2)_{\leq}$ is a matroid base system. Let r' be the rank function of $\mathcal{B}(M) \cap (A, 2)_{\leq}$. Note that $r'(X) \leq r(X)$ always holds. We separate the cases according to (3-1), (3-2) and (3-3) to show submodularity $r'(A) + r'(F) \geq r'(A \cup F) + r'(A \cap F)$ for any facet-defining flat F of rank 2. In the case (3-1), $r'(F) = r(F) = 2$ since $r'(F - A) = 2$. So the submodularity holds in this case. In the case (3-2), the submodularity holds trivially. To consider the case (3-3), we assume $F \cup A = E$. Since $r'(A) = 2$ and $r'(F \cap A) \leq r(F \cap A)$, it suffices to show $r(F \cap A) \leq 1$. When $r(F \cap A) = 2$, there exists $B \in \mathcal{B}(M)$ such that $|B \cap (F \cap A)| = 2$. Since $|B| = 3$ and $F \cup A = E$, $|F \cap B| = 3$ or $|A \cap B| = 3$, which contradicts $r(F) = 2$ and $r(A) = 2$. Therefore $r(F \cap A) = 1$. We have completed the case (3-3).

By Theorem 2, $\mathcal{B}(M) \cap (A, 2)_{\leq}$ is a matroid base system. So $\mathcal{B}(M) \cap (A, 2)_{=}$ is a matroid base system.

By Condition (1) and the simpleness of the matroid, there exists a base $B \in \mathcal{B}(M)$ such that $|B \cap A| \leq 1$. Therefore $\mathcal{B}(M) \cap (A, 2)_{<}$ is not empty. By Condition (2), $\mathcal{B}(M) \cap (A, 2)_{>}$ is not empty. Therefore, $\mathcal{B}(M)$ is 2-decomposable by Theorem 22.

We consider the converse direction. Assume that there exists a flat A of rank 2 and which satisfies neither of (3-1), (3-2), nor (3-3). Then there exists a facet-defining inequality $(F, 2)_{\leq}$ such that $|A \cap F| > 1$, $F - A \neq \emptyset$, $A \cup F \neq E$. Since $F - A \neq \emptyset$ and A is a flat of rank 2, we have $r'(A \cup F) = 3$. Since $|A \cap F| > 1$ and $A \cup F \neq E$, we have $r'(A \cap F) \geq 1$. Therefore $\mathcal{B}(M) \cap (A, 2)_{\leq}$ does not satisfy submodularity $r'(A) + r'(F) \geq r'(A \cup F) + r'(A \cap F)$. When the rank of A is 2, $\mathcal{B}(M) \cap (A, 2)_{>}$ is empty. When $|A^c| \leq 1$, $\mathcal{B}(M) \cap (A, 2)_{\leq}$ is not connected. Therefore when one of Conditions (1),(2) and (3) is not satisfied, $\mathcal{B}(M)$ is not 2-decomposable. \square

Corollary 30. *Consider a connected simple matroid base system $\mathcal{B}(M)$ of rank 3. If there exist $x, y \in E$ with $x \neq y$ such that $\{F \in \mathcal{F}^2(M) | x \in F\} = \{F \in \mathcal{F}^2(M) | y \in F\}$ and the rank of $\{x, y\}^c$ is 3, then it is 2-decomposable by $(\{x, y\}, 1)_{=}$ where $\mathcal{F}^2(M)$ denotes the set of facet-defining flats of rank 2.*

Proof. It suffices to show that $\{x, y\}^c$ satisfies Conditions (1), (2), and (3) of Theorem 29. Condition (1) follows from $|\{x, y\}| = 2$. Condition (2) follows from the assumption $r(\{x, y\}^c) = 3$. We check the Condition (3). Note that $|F \cap \{x, y\}| \neq 1$ holds for a facet-defining flat F of rank 2 by the assumption. Condition (3-3) $F \cup \{x, y\}^c = E$ holds when a facet-defining flat F of rank 2 satisfies $F \supseteq \{x, y\}$. Condition (3-2) holds when a facet-defining flat F of rank 2 satisfies $F \cap \{x, y\} = \emptyset$. \square

Example 31. *Let $E = \{a, b, c, d, e, f\}$. Consider the following matroid base system.*

$$\mathcal{B}(M) = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e, f\}, 2)_{\leq}.$$

Since $\{e, f\}$ satisfies the conditions in Corollary 30, $\mathcal{B}(M)$ is 2-decomposable by $(\{e, f\}, 1)_{=}$ into $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$.

$$\mathcal{B}(M_1) = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e, f\}, 2)_{\leq} \cap (\{e, f\}, 1)_{\leq},$$

$$\mathcal{B}(M_2) = (E, 3)_{=} \cap (\{c, d, e, f\}, 2)_{\leq} \cap (\{a, b, c, d\}, 2)_{\leq}.$$

Corollary 32. *Consider a connected simple matroid base system of rank 3 with $|E| \geq 5$. Define the graph on E so that $\{x, y\}$ is an edge of the graph if and only if there exists no facet-defining flat of rank 2 including $\{x, y\}$. If the graph has a 3-cycle on $\{x, y, z\}$, the matroid base system is 2-decomposable by $(\{x, y, z\}, 2)_{=}$.*

Proof. Let $\{x, y, z\}$ have such a 3-cycle. We check that $A = \{x, y, z\}$ satisfies the conditions in Theorem 29. Condition (1) is satisfied because of $|E| \geq 5$. Conditions (2) and (3-1) follow from that $\{x, y, z\}$ intersects any facet-defining flat of rank 2 in at most 1 element. \square

Example 33. *Let $E = \{a, b, c, d, e, f\}$. Consider the following matroid base system.*

$$\mathcal{B}(M) = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e\}, 2)_{\leq} \cap (\{e, f, a\}, 2)_{\leq}.$$

Since each of $\{b, d\}, \{d, f\}, \{f, b\}$ is contained in no facet-defining flat of rank 2, $\{b, d, f\}$ contains a 3-cycle satisfying the conditions in Corollary 32. Therefore the matroid base system is 2-decomposable by $(\{b, d, f\}, 2)_{=}$.

$$\mathcal{B}(M_1) = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{c, d, e\}, 2)_{\leq} \cap (\{e, f, a\}, 2)_{\leq} \cap (\{b, d, f\}, 2)_{\leq}.$$

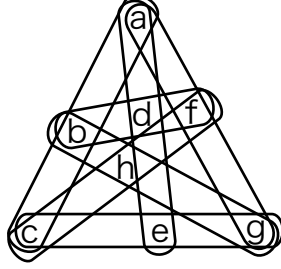


Figure 4: The flats in the matroid in Example 35

$$\mathcal{B}(M_2) = (E, 3)_{=} \cap (\{a, c\}, 1)_{\leq}.$$

Corollary 34. *For a connected simple matroid base system of rank 3, if there exist a flat A of rank 2 and $x \in E - \bigcup\{F \in \mathcal{F}^2 | A \cap F \neq \emptyset\} - A$ and $|E - A| \geq 3$, the matroid base system is 2-decomposable by $(A \cup x, 2)_{=}$, where \mathcal{F}^2 is the set of the facet-defining flats of rank 2.*

Proof. We have only to show Conditions (1), (2), and (3) in Theorem 29. Condition (1) follows from $|E - A| \geq 3$. Condition (2) $r(A \cup x) = 3$ follows from that A is a flat of rank 2 and $x \notin A$. For every facet-defining flat F of rank 2, $|F \cap (A \cup x)| = |F \cap A|$ when $A \cap F \neq \emptyset$ since $x \notin F$. $r(F \cap A) \leq 1$ by the submodularity. $|F \cap A| \leq 1$ since a matroid is simple. When $A \cap F = \emptyset$, $|F \cap (A \cup x)| \leq 1$ holds. Therefore we have (3-1) $|F \cap (A \cup x)| \leq 1$. \square

Example 35. *Let $E = \{a, b, c, d, e, f, g, h\}$. The following example is a matroid base system which is 2-decomposable by Corollary 34.*

$$\mathcal{B}(M) = (E, 3)_{=} \cap (abc, 2)_{\leq} \cap (ade, 2)_{\leq} \cap (afg, 2)_{\leq} \cap (bdf, 2)_{\leq} \cap (ceh, 2)_{\leq} \cap (bgh, 2)_{\leq} \cap (cfh, 2)_{\leq}.$$

This matroid is depicted in Figure 4. Corollary 32 cannot apply to this matroid base system. However, since $\{a, d, e\}$ and h satisfy the conditions in Corollary 34, it is 2-decomposable by $(\{a, d, e, h\}, 2)_{=}$.

The 2-decomposability of a connected simple matroid base system of small size is always decided by Corollaries 30, 32 and 34. However, that of some connected simple matroid base system cannot be decided by the above corollaries as in the next example.

Example 36. *Let $E = \{a, b, c, d, e, f, g, h, i, j, k, l\}$. Consider the following matroid base system on E whose facet-defining flats of rank 2 are*

$$adgi, bcei, abhj, cdfj, acl, bdl, egl, fhl, ijl, aek, bfk, cgk, dhk.$$

Since the matroid has too many facet-defining flats, we use two figures to represent them. The first figure in Figure 5 indicates the facet-defining flats of rank 2 containing i or j . The second figure indicates the facet-defining flats of rank 2 containing k or l . The matroid has facet-defining flat $\{i, j, l\}$ other than those depicted in the figures.

There exists no pair of elements as in Corollary 30. There exists no 3-cycle as in Corollary 32. There exists no facet-defining flat as in Corollary 34. However, by Theorem 29, it is 2-decomposable by $(\{e, f, g, h, l\}, 2)_{=}$.

4.3 Included matroids and 3-partitions

Since we handle two or more matroids simultaneously, we name them to distinguish. We call a connected simple matroid base system that we consider as a whole matroid an *original matroid base system*. In this

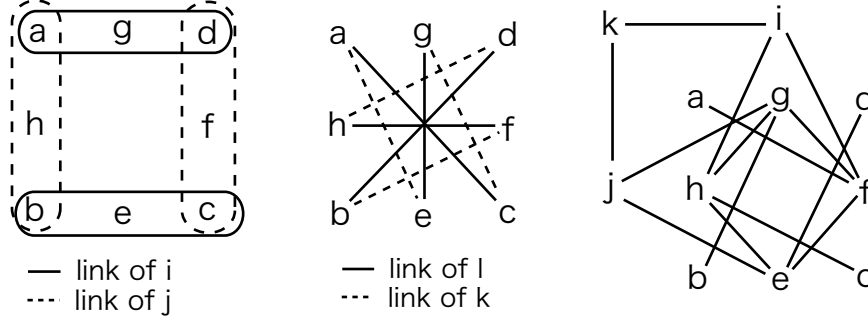


Figure 5: The first two figures show the facet-defining flats. The last figure shows the graph defined in Corollary 32

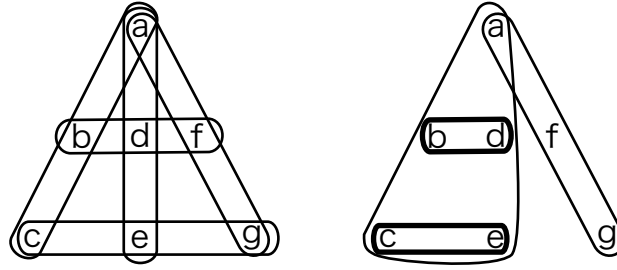


Figure 6: The original matroid M and an included matroid M_1 in Example 37

section, any original matroid base system is assumed to be of rank 3. A facet-defining flat of the original matroid base system is called an *original facet-defining flat*.

We fix an original matroid base system of rank 3. A connected matroid base system that is properly included in the original matroid base system is called an *included matroid base system*. Note that an included matroid base system may not be simple. A facet-defining flat of an included matroid base system that is not an original facet is called a *non-original facet-defining flat*. For an original matroid base system $\mathcal{B}(M)$ with its rank function r , and its included matroid base system $\mathcal{B}(M')$ with its rank function r' , $r'(A) \leq r(A)$ holds for any $A \subseteq E$ by Lemma 15.

It is known that any binary matroid base system cannot include any included matroid base system as shown in Theorem 48.

Example 37. The next example is a matroid base system on E of size 7 that is neither binary nor 2-decomposable but not minimal with respect to the weak-map order. Let $E = \{a, b, c, d, e, f, g\}$.

$$\mathcal{B}(M) = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{a, d, e\}, 2)_{\leq} \cap (\{a, f, g\}, 2)_{\leq} \cap (\{b, d, f\}, 2)_{\leq} \cap (\{c, e, g\}, 2)_{\leq}.$$

Since any two facet-defining flats of rank 2 intersect in at most one element. It is not binary since $M/\{c\} \setminus \{b, g\} = U_{2,4}$. However, it is not 2-decomposable by Theorem 29.

$$\mathcal{B}(M_1) = (E, 3)_{=} \cap (\{a, b, c, d, e\}, 2)_{\leq} \cap (\{a, f, g\}, 2)_{\leq} \cap (\{b, d\}, 1)_{\leq} \cap (\{c, e\}, 1)_{\leq}$$

is an included matroid base system of M , shown in Figure 6. For example, $(\{a, b, c, d, e\}, 2)_{\leq}$ is a non-original facet-defining flat, and $(\{a, f, g\}, 2)_{\leq}$ is an original facet-defining flat.

Definition 38. For a connected simple matroid base system $\mathcal{B}(M)$ of rank 3 with its rank function r , a partition $\{A_1, A_2, A_3\}$ of E is called a 3-partition in M if the following conditions are satisfied.

- $|A_1|, |A_2|, |A_3| \geq 2$,
- there exists no facet-defining flat of rank 2 that intersects all of A_1, A_2, A_3 ,
- $r(A_1 \cup A_2) = r(A_2 \cup A_3) = r(A_3 \cup A_1) = 3$.

A 3-partition $\{A_1, A_2, A_3\}$ corresponds to a matroid $(A_1, 1)_= \cap (A_2, 1)_= \cap (A_3, 1)_=$. When the matroid corresponding to a 3-partition in M is a ridge of its included matroid M' , we say that $\mathcal{B}(M')$ has a 3-partition in M .

Lemma 39. Consider a connected simple matroid base system $\mathcal{B}(M)$ of rank 3 which is not 2-decomposable and has an included matroid base system $\mathcal{B}(M')$. Then $\mathcal{B}(M')$ has a 3-partition $\{A_1, A_2, A_3\}$ in M such that $(A_1, 1)_\leq$ and $(A_1 \cup A_2, 2)_\leq$ are non-original facet-defining flats.

Proof. Let r' be the rank function of M' . Since the matroid base system $\mathcal{B}(M)$ is not 2-decomposable, $\mathcal{B}(M')$ has at least two non-original facet-defining flats.

Consider the case where every non-original facet-defining flat of $\mathcal{B}(M')$ has rank 1. Let $(A_1, 1)_\leq$ be a non-original facet-defining flat of M' . Then any other non-original facet-defining inequality $(A, 1)_\leq$ of $\mathcal{B}(M')$ is disjoint from A_1 on E since M' is connected. Then $\mathcal{B}(M) \cap (A_1, 1)_\leq$ is a matroid base system since $\mathcal{I}(M) \cap (A_1, 1)_\leq$ is a matroid independence system by Theorem 2. By Theorem 22, $\mathcal{B}(M)$ is 2-decomposable, a contradiction.

So we can assume that $\mathcal{B}(M')$ has a non-original facet-defining inequality $(F_1, 2)_\leq$.

Since the matroid base system $\mathcal{B}(M)$ cannot be 2-decomposable by non-original facet-defining equality $(F_1, 2)_=$, $\mathcal{B}(M) \cap (F_1, 2)_\leq$ is not a matroid base system. Therefore $\mathcal{B}(M')$ has another non-original facet-defining inequality than $(F_1, 2)_\leq$.

We show that there exists a non-original facet-defining inequality $(A_1, 1)_\leq$ such that $A_1 \subseteq F$. On the contrary, suppose that there exists no such inequality. If any other non-original facet-defining inequality than $(F_1, 2)_\leq$ is either of the following three types (a), (b) and (c), it is easy to show that the submodularity $r''(F_1) + r''(F_2) \geq r''(F_1 \cap F_2) + r''(F_1 \cup F_2)$ of $\mathcal{B}(M) \cap (F_1, 2)_\leq$ holds where r'' is the rank function of $\mathcal{B}(M) \cap (F_1, 2)_\leq$.

(a) non-original facet-defining inequality $(A, 1)_\leq$ with $A \cap F_1 = \emptyset$.

(b) non-original facet-defining inequality $(F, 2)_\leq$ with $|F \cap F_1| \leq 1$.

(c) non-original facet-defining inequality $(F, 2)_\leq$ with $F \cup F_1 = E$.

Then $\mathcal{B}(M)$ is 2-decomposable by $(F_1, 2)_=$ at that time by Theorem 22. Therefore there exists a flat F_1 of rank 2 such that $|F \cap F_1| \geq 2$ and $F \cup F_1 \neq E$. Then $F \cap F_1$ is a flat of rank 1 by Corollary 26. By letting $A_1 = F \cap F_1$, $(A_1, 1)_\leq$ is a non-original facet-defining inequality on $\mathcal{B}(M')$ such that $A_1 \subseteq F_1$ and $|A_1| \geq 2$, a contradiction.

We show that $\{A_1, A_2, A_3\}$ is a 3-partition in M where $A_2 = F_1 - A_1$ and $A_3 = F_1^c$.

Since $(A_1 \cup A_2, 2)_\leq$ is a non-original facet-defining inequality, $\mathcal{B}(M) \cap (A_1 \cup A_2, 2)_>$ is non-empty. Therefore $A_1 \cup A_2$ has rank 3 on the matroid base system $\mathcal{B}(M)$. If $A_1 \cup A_3$ has rank 2 on M , it contradicts Corollary 26 since $(A_1, 1)_\leq$ is a non-original facet-defining inequality and $(A_1 \cup A_2) \cup (A_1 \cup A_3) = E$. So $r(A_1 \cup A_3) = 3$. Since the rank of A_1 is 1 and the included matroid M' is connected, $A_2 \cup A_3$ has rank 3 on the matroid M .

Since $(A_1, 1)_\leq$ is a non-original facet-defining inequality, we have $|A_1| \geq 2$. Since $(A_1 \cup A_2, 2)_\leq$ is a non-original facet-defining inequality, $|A_2| \geq 2$. Since $|A_3| = 1$ contradicts the connectivity of the matroid M , we have $|A_3| \geq 2$.

$\mathcal{B}(M) \cap (A_1, 1)_= \cap (A_1 \cup A_2, 2)_= = \mathcal{B}(M) \cap (A_1, 1)_= \cap (A_2, 1)_= \cap (A_3, 1)_=$ includes a matroid base system of rank 3 as a ridge of $\mathcal{B}(M')$. There exists no loopless matroid included in $\mathcal{B}(M) \cap (A_1, 1)_= \cap (A_2, 1)_= \cap (A_3, 1)_=$ for any facet-defining flat F of rank 2 that intersects all of A_1, A_2, A_3 . Note that the ridge of M' defined by $(A_1, 1)_=$ and $(A_1 \cup A_2, 2)_=$ cannot have any loop by considering its dimension. Therefore there exists no original facet-defining flat of rank 2 that intersects all of A_1, A_2, A_3 . Therefore $\{A_1, A_2, A_3\}$ is a 3-partition in M . \square

By this lemma, a connected simple matroid of rank 3 whose base system has an included matroid base system has a 3-partition $\{A_1, A_2, A_3\}$ in M and therefore the non-connected matroid base system $(A_1, 1)_= \cap (A_2, 1)_= \cap (A_3, 1)_=$ is included in the original matroid base system, which is a ridge of the included matroid base system.

Let $e(k)$ be $\binom{k}{2}$, which is the number of edges of the complete graph of size k . We define function f as follows. Let $f(2k) = k \times (k-1)$ for even number $2k$ and $f(2k+1) = k^2$ for odd number $2k+1$. $f(n)$ is equal to the minimum number of $e(n_1) + e(n_2)$ so that $n = n_1 + n_2$. In fact, $f(2k) = 2 \times e(k) = 2 \times \frac{1}{2}k \times (k-1)$ and $f(2k+1) = e(k) + e(k+1) = \frac{1}{2}k \times (k-1) + \frac{1}{2}(k+1) \times (k+1) = k^2$.

Lemma 40. *Consider a connected simple matroid M of rank 3. When $\{A_1, A_2, A_3\}$ is a 3-partition in M , $e(|A_1|) + e(|A_2|) + e(|A_3|)$ is equal to or more than the sum of $f(|F|)$ over all facet-defining flats F of rank 2.*

Proof. Consider the three complete graphs such that the sets of vertices are A_1, A_2, A_3 . Then the sum of the numbers of their edges are $e(|A_1|) + e(|A_2|) + e(|A_3|)$. Any original facet-defining flat F cannot intersect all of A_1, A_2, A_3 by the definition of a 3-partition. For each A_i , each pair $\{x, y\}$ of $x, y \in A_i$ is included in at most one original facet-defining flat of rank 2 by Lemma 24 and the simpleness of M . Each facet-defining flat F is included in one of $A_1 \cup A_2$, $A_2 \cup A_3$, or $A_3 \cup A_1$. Therefore the number of edges included in each F is at least $f(|F|)$. \square

4.4 The graph induced from original facets

In this subsection, we introduce a graph to investigate the existence of an included matroid.

Definition 41. *Consider a connected simple matroid M of rank 3 as an original matroid. For any disjoint sets A_1 and A_2 on E , we define the graph $g(A_1, A_2)$ as follows. The set of vertices of $g(A_1, A_2)$ is A_2 . For $x, y \in A_2$, $\{x, y\}$ is an edge if and only if $\mathcal{B}(M)$ has a facet-defining flat F of rank 2 such that $\{x, y\} \subseteq F$ and $|F \cap A_1| \geq 1$.*

The graph $g(A_1, A_2)$, which is defined on A_2 , is said to be *connected* if A_2 is a connected component of the graph.

Lemma 42. *For a connected simple matroid M of rank 3 and a 3-partition $\{A_1, A_2, A_3\}$ in M , $g(A_1, A_3)$ and $g(A_2, A_3)$ have no common edge.*

Proof. Suppose that $\{x, y\}$ is a common edge of $g(A_1, A_3)$ and $g(A_2, A_3)$. By the definition of the graph, there exists a facet-defining flat F_1 of rank 2 including $\{x, y\}$ and intersecting A_1 . Moreover there exists a facet-defining flat F_2 of rank 2 including $\{x, y\}$ and intersecting A_2 . By the definition of a 3-partition, there exists no facet-defining flat intersecting all of A_1, A_2 , and A_3 . Therefore F_1 and F_2 are distinct. The intersection of two distinct facet-defining flats F_1 and F_2 of rank 2 has at most rank 1 by Lemma 24. Since M is simple, we have $|F_1 \cap F_2| \leq 1$. It contradicts that both F_1 and F_2 include $\{x, y\}$. \square

Given an original matroid $\mathcal{B}(M)$, how do we find an included matroid base system? By using Lemma 39, we can try to find a 3-partition in M . Consider the original matroid and some additional inequalities as non-original facet-defining inequalities obtained from the 3-partition. To make an included matroid base system, what inequalities we should add further to such a matroid base system as restrictions?

When it satisfies $(F_1, 1)_\leq$ and $(F_2, 2)_\leq$ with neither $F_1 \subseteq F_2$ nor $F_1 \cap F_2 = \emptyset$, assume that it satisfies $(F_1 \cup F_2, 2)_\leq$.

For $(F_1, 2)_\leq$ and $(F_2, 2)_\leq$, when $|F_1 \cap F_2| \geq 2$ and $r(F_1 \cap F_2) = 2$, we have two ways to hold submodularity $r(F_1) + r(F_2) \geq r(F_1 \cup F_2) + r(F_1 \cap F_2)$: (1) add $(F_1 \cap F_2, 1)_\leq$ as a restriction, (2) add $(F_1 \cup F_2, 2)_\leq$ as a restriction.

Note that we cannot add inequalities when they contradict that $(A_1 \cup A_2, 2)_\leq$ appeared in Lemma 39 is a facet-defining inequality. When F_1 or F_2 is such a flat, the additional inequality should be of type (1).

Lemma 43. *Consider a connected simple matroid M of rank 3 as an original matroid. Assume that $\mathcal{B}(M)$ has an included matroid base system $\mathcal{B}(M')$ such that A is a flat of rank 1. Let $C \subseteq E$ be a (non-empty) connected vertices of $g(A, B)$ where B is a non-empty set disjoint from A . Then $A \cup C$ has rank 2 on M' .*

Proof. We use an induction on the size of connected vertices C of $g(A, B)$.

The case of $|C| = 1$ follows from that A is a flat on M' . Consider the case of $C = \{x, y\}$ where $\{x, y\}$ is an edge of $g(A, B)$. By the definition of $g(A, B)$, there exists $z \in A$ such that $\{x, y, z\}$ has rank 2 on M . Since A is a flat of rank 1 on M' , $A \cup \{x, y\}$ has rank 2 on M' by submodularity.

Next we show the inductive step of the proof. Assume that C is connected and $C \cup x$ is connected on the graph $g(A, B)$. By the induction hypothesis, $A \cup C$ has rank 2 on M' . Let $\{x, y\}$ be an edge of $g(A, B)$ with $y \in C$. By the induction hypothesis, $A \cup \{x, y\}$ has rank 2 on M' . Since A is a flat of rank 1 on M' , $(A \cup C) \cap (A \cup \{x, y\}) = A \cup y$ has rank 2 on M' . By submodularity $r'(A \cup C) + r'(A \cup \{x, y\}) \geq r'(A \cup y) + r'(A \cup C \cup x)$ on M' , $A \cup C \cup x$ has rank 2 on M' . \square

Lemma 44. *Consider a connected simple matroid M of rank 3 as an original matroid. Assume that $\mathcal{B}(M)$ has an included matroid base system $\mathcal{B}(M')$ that has a flat A of rank 2. Then every connected component of $g(B, A)$ has rank 1 on M' where B is a non-empty set disjoint from A .*

Proof. When $\{x\}$ is contained in no edges in $g(B, A)$, the statement holds since M' is connected and therefore loopless. Let $\{x, y\}$ be an edge of $g(B, A)$. By the definition of $g(B, A)$, there exists $z \in B$ such that $\{x, y, z\}$ has rank 2 on the included matroid M' . Let r' be the rank function of the included matroid M' . Note that $r'(X) \leq r(X)$ for any $X \subseteq E$ by Lemma 15. Since A is a flat of rank 2 on M' , we have $r'(A \cup z) = 3$. Therefore, by submodularity $r'(\{x, y, z\}) + r'(A) \geq r'(A \cup z) + r'(\{x, y\})$, we have $r'(\{x, y\}) = 1$. Since the parallel relation on a matroid is an equivalence relation, the connected component including $\{x, y\}$ has rank 1 on M' . \square

Example 45. *Consider a matroid M_1 shown in Example 37. $g(\{b, d\}, \{b, d\}^c)$ has edges $\{a, c\}$ and $\{a, e\}$. Therefore, its connected components are $\{a, c, e\}$, $\{g\}$, and $\{f\}$. When there exists an included matroid base system with a facet-defining inequality $(\{b, d\}, 1)_\leq$, by Lemma 43, $r'(\{b, d, a, c, e\}) = 2$. By Lemma 44, $r'(\{c, e\}) = 1$. In fact, $\mathcal{B}(M_1)$ has an included matroid base system $\mathcal{B}(M_2)$ shown in Example 37.*

4.5 Decomposition of a matroid base system of rank 3

In this subsection, we consider decomposition problem of a connected simple matroid base system of rank 3. A facet of a matroid base system is also a matroid base system. Note that the matroid base system $\mathcal{B}(M) \cap (A, r(A))_\leq$ defined by a facet-defining inequality $(A, r(A))_\leq$ of rank 1 or 2 has also rank 3. We call the matroid base system defined as a facet the *matroid base system on the facet*. By Theorem 13, the matroid base system defined as a facet has two connected components: one connected component is a matroid base system of rank 1, the other connected component is a matroid base system of rank 2.

Consider a decomposable connected matroid base system $\mathcal{B}(M)$ of rank 3. $\mathcal{B}(M)$ has an included matroid base system $\mathcal{B}(M')$ with a rank function r' in the decomposition. For a non-original facet-defining inequality $(A, r'(A))_\leq$ of $\mathcal{B}(M')$, M has another included matroid base system that has the facet defined by the inverse inequality $(A, r'(A))_\geq$ by the definition of the decomposition. We call such an included matroid base system an *included matroid base system on the other side of the facet*.

For two included matroid base systems $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ that have a common facet, the intersection $\mathcal{B}(M_1) \cap \mathcal{B}(M_2)$ is the matroid base system on the common facet. When a matroid M and an included matroid base system $\mathcal{B}(M')$ of M are given, the matroid base system $\mathcal{B}(M') \cap (A, r'(A))_\leq$ defined as a facet narrows down possible candidates for included matroid base systems on the other side of the facet since any included matroid base system on the other side must have $\mathcal{B}(M') \cap (A, r'(A))_\leq$ as a facet. Note that the rank of a non-original facet-defining flat is 1 or 2 on M' .

Example 46. *We consider the matroid shown in Example 37 again. This matroid base system $\mathcal{B}(M)$ is decomposed into four included matroid base systems M_1, M_2, M_3 , and M_4 as follows. Since $\mathcal{B}(M_1)$ has three non-original facet-defining flats $(\{a, b, c, d, e\}, 2)_\leq$, $(\{b, d\}, 1)_\leq$ and $(\{c, e\}, 1)_\leq$, we try to find*

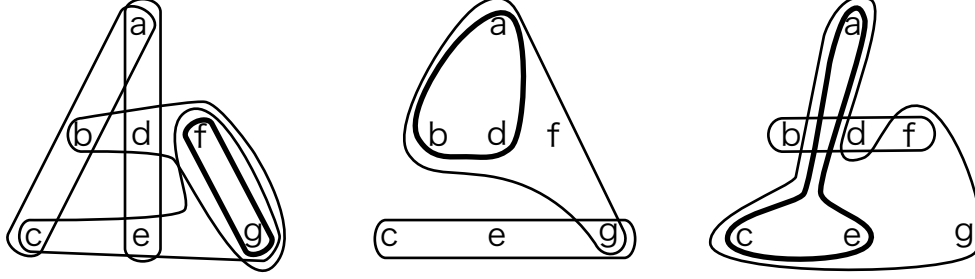


Figure 7: Included matroids M_2 , M_3 and M_4

an included matroid base system on the other side of each of the three facets. The matroid M_{12} on $(\{a, b, c, d, e\}, 2)_{\leq}$ is obtained from imposing $(\{f, g\}, 1)_{\leq}$ on M_1 since $(E, 3)_{=} \cap (\{a, b, c, d, e\}, 2)_{=} = (E, 3)_{=} \cap (\{a, b, c, d, e\}, 2)_{\leq} \cap (\{f, g\}, 1)_{\leq}$.

$$\mathcal{B}(M_{12}) = (E, 3)_{=} \cap (\{a, b, c, d, e\}, 2)_{\leq} \cap (\{b, d\}, 1)_{\leq} \cap (\{c, e\}, 1)_{\leq} \cap (\{f, g\}, 1)_{\leq}.$$

The included matroid base system $\mathcal{B}(M_2)$ on the other side of the facet $\mathcal{B}(M_{12})$ should satisfy the following conditions.

- $\mathcal{B}(M_2)$ is included in $\mathcal{B}(M)$,
- $(\{f, g\}, 1)_{\leq}$ is a non-original facet-defining inequality of $\mathcal{B}(M_2)$,
- $\mathcal{B}(M_{12})$ is the matroid base system on $(\{f, g\}, 1)_{=}$ as a facet of $\mathcal{B}(M_2)$.

Note that $\mathcal{B}(M_{12}) = \mathcal{B}(M_1) \cap \mathcal{B}(M_2)$. So we have

$$\mathcal{B}(M_2) = (E, 3)_{=} \cap (\{a, b, c\}, 2)_{\leq} \cap (\{a, d, e\}, 2)_{\leq} \cap (\{c, e, f, g\}, 2)_{\leq} \cap (\{f, g\}, 1)_{\leq}.$$

Similarly, we can find the included matroid base systems in the decomposition other than $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$.

$$\mathcal{B}(M_3) = (E, 3)_{=} \cap (\{a, b, d, f, g\}, 2)_{\leq} \cap (\{c, e, g\}, 2)_{\leq} \cap (\{a, b, d\}, 1)_{\leq},$$

$$\mathcal{B}(M_4) = (E, 3)_{=} \cap (\{a, c, e, f, g\}, 2)_{\leq} \cap (\{b, d, f\}, 2)_{\leq} \cap (\{a, c, e\}, 1)_{\leq}.$$

Among these four matroids, the following are the pairs of matroids whose base systems have a common facet.

$$\{M_1, M_2\}, \{M_1, M_3\}, \{M_1, M_4\}, \{M_2, M_3\}, \{M_2, M_4\}.$$

Since $\mathcal{B}(M_3) \cap \mathcal{B}(M_4) = (\{a\}, 0)_{=} \cap (\{b, d\}, 1)_{=} \cap (\{c, e\}, 1)_{=} \cap (\{f, g\}, 1)_{=}$, $\mathcal{B}(M_3)$ and $\mathcal{B}(M_4)$ do not have any common facet.

Lemma 47. Consider a connected simple matroid M of rank 3 as an original matroid that has a 3-partition $\{A_1, A_2, A_3\}$. Assume that an included matroid base system $\mathcal{B}(M_1)$ has non-original facet-defining inequalities $(A_1, 1)_{\leq}$ and $(A_1 \cup A_2, 2)_{\leq}$. Assume that there exists a decomposition using $\mathcal{B}(M_1)$. Then there exists an included matroid base system which has non-original facet-defining inequalities $(A_3, 1)_{\leq}$ and $(A_3 \cup A_1, 2)_{\leq}$. Moreover, there exists another included matroid base system which has non-original facet-defining inequalities $(A_2, 1)_{\leq}$ and $(A_2 \cup A_3, 2)_{\leq}$.

Proof. Since $\mathcal{B}(M)$ has a decomposition using $\mathcal{B}(M_1)$, there exists an included matroid base system on the other side of $(A_1, 1)_{=}$, and there exists an included matroid base system on the other side of $(A_1 \cup A_2, 2)_{=}$.

By considering the matroid base system $\mathcal{B}(M_1) \cap (A_1 \cup A_2, 2)_{=}$ as a facet of $\mathcal{B}(M)$, the included matroid base system $\mathcal{B}(M_2)$ on the other side of $(A_1 \cup A_2, 2)_{\leq}$ has non-original facet-defining inequality

$(A_3, 1)_\leq$. Because $(A_1, 1)_= \cap (A_2, 1)_= \cap (A_3, 1)_=$ is a ridge of the facet $\mathcal{B}(M_1) \cap (A_1 \cup A_2, 2)_=$, another facet-defining equality should contain this ridge. Therefore, the included matroid base system $\mathcal{B}(M_2)$ has non-original facet-defining inequality $(A_3 \cup A_1, 2)_\leq$.

By considering the matroid base system $\mathcal{B}(M_1) \cap (A_1, 1)_=$ on the facet, the included matroid base system on the other side of $(A_1, 1)_\leq$ has non-original facet-defining inequalities $(A_2 \cup A_3, 2)_\leq$ and $(A_2, 1)_\leq$. \square

Consider small matroid base systems of rank 3 which are neither binary nor 2-decomposable by our computation. There exists no such matroid of size 6. There exist two such matroids of size 7. There exist five such matroids of size 8.

5 Main results about matroid base systems

This section is the main part of this paper and consists of two subsections. In Section 5.1, we classify the matroid base systems according to their decomposability. In Section 5.2, we give a counterexample to the conjecture proposed by Lucas [10].

5.1 A classification of matroid base systems

In this subsection, we classify the matroid base systems according to their decomposability.

Lucas proved the following theorem.

Theorem 48. (Lucas [10]) *There exists no connected binary matroid whose base system includes that of another connected matroid on the same ground set.*

In other words, a connected binary matroid is minimal in all the connected matroids on the same ground set with respect to weak-map order of the same rank.

We have the next corollary from Lemma 18.

Corollary 49. *No connected matroid base system which is minimal with respect to weak-map order in all the connected matroids is decomposable.*

We classify matroid base systems into five types (a), (b), (c), (d), and (e) with respect to their decomposability and weak-map order.

(a) Binary matroids.

A binary matroid is minimal in the connected matroids with respect to inclusion by Theorem 48. Therefore its base system is not decomposable by Corollary 49.

(b) Non-binary but minimal matroids in the connected matroids with respect to inclusion.

A connected matroid base system of rank 4 which does not include any included matroid base system is known (Lucas [10]). We give a connected matroid of rank 3 which does not include any included matroid base system.

Example 50. Let $E = \{a, b, c, d, e, f, g, h\}$. The following connected simple matroid base system is an example of a matroid whose base system does not include any included matroid base system.

$$\mathcal{B}(M) = (E, 3)_= \cap (afd, 2)_\leq \cap (ebh, 2)_\leq \cap (abg, 2)_\leq \cap (efc, 2)_\leq \cap (egd, 2)_\leq \cap (ach, 2)_\leq \cap (bcd, 2)_\leq \cap (fgh, 2)_\leq.$$

These flats define a matroid because any two flats of rank 2 intersect in at most one element. This matroid is shown in Figure 8 where each line represents a flat of rank 2.

Moreover, it is not binary since $M/\{a\} \setminus \{d, g, h\} = U_{2,4}$. We prove that it does not include any included matroid base system in Theorem 52.

Lemma 51. *The matroid base system shown in Example 50 has no 3-partition.*

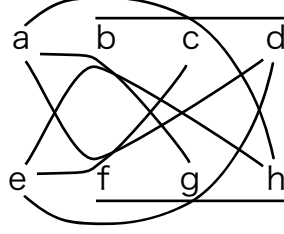


Figure 8: A matroid M which does not include any included matroid

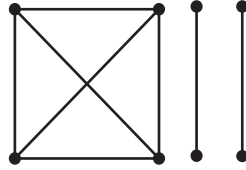


Figure 9: Complete graphs of A_1, A_2 and A_3

Proof. By Lemma 40, no 3-partition is of size $(3, 3, 2)$ because $e(3) + e(3) + e(2) = 7 < f(3) \times 8 = 8$. Therefore the size of the 3-partition $\{A_1, A_2, A_3\}$ is $(|A_1|, |A_2|, |A_3|) = (4, 2, 2)$. Since any original facet-defining flat of rank 2 cannot intersect all of A_1, A_2 and A_3 , any original facet-defining flat of rank 2 intersects one of A_1, A_2 or A_3 in at least two elements. Note that any original facet-defining flats of rank 2 intersect in at most one element by Lemma 24. Since $e(4) + e(2) + e(2) = 8$, each edge of the graph in Figure 9 is included in exactly one original facet-defining flat of rank 2 by a similar argument to the proof of Lemma 40.

By Lemma 42, $g(A_2, A_1)$ and $g(A_3, A_1)$ have no common edge. Since the number of the edges in the complete graph on A_1 is 6, the maximum of $g(A_2, A_1) + g(A_3, A_1)$ is 6. Since the number of the facet-defining flats of rank 2 is 8, this implies that $A_2 \cup A_3$ includes at least two original facet-defining flats of rank 2, which contradicts that two facet-defining flats intersect in at most one element by Lemma 24. \square

Theorem 52. *The matroid base system shown in Example 50 does not include any included matroid base system.*

Proof. Suppose that the matroid base system $\mathcal{B}(M)$ has an included matroid base system $\mathcal{B}(M')$. Note that this matroid base system $\mathcal{B}(M)$ has 8 facet-defining flats of rank 2 and is not 2-decomposable. By Lemma 39, $\mathcal{B}(M')$ has a 3-partition in M , which contradicts Lemma 51. \square

(c) Non-binary and non-minimal but indecomposable matroids.
We give an example of type (c) of size 8 in Example 53.

Example 53. *The next example is an indecomposable matroid base system which includes an included matroid base system.*

Let $E = \{a, b, c, d, e, f, g, h, i\}$.

$$\mathcal{B}(M) = (E, 3)_{=} \cap (bdfh, 2)_{\leq} \cap (abc, 2)_{\leq} \cap (ade, 2)_{\leq} \cap (afg, 2)_{\leq} \cap (ahi, 2)_{\leq} \cap (bgi, 2)_{\leq} \cap (cdi, 2)_{\leq} \cap (cef, 2)_{\leq} \cap (egh, 2)_{\leq}.$$

The matroid M is depicted in the left figure of Figure 10 where each line represents a facet-defining flat of rank 2.

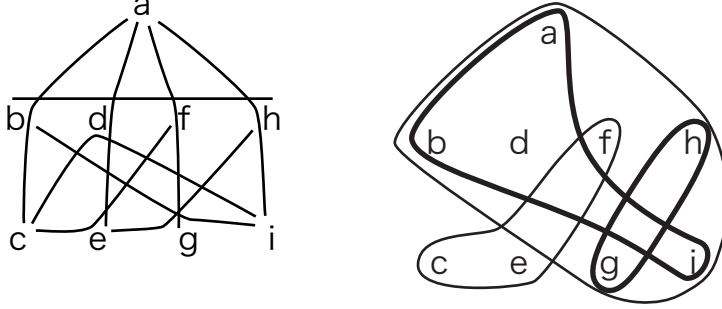


Figure 10: Indecomposable original matroid M and its included matroid M_1

This matroid base system includes the following included matroid base system shown in Figure 10.

$$\mathcal{B}(M_1) = (E, 3)_{=} \cap (\{a, b, d, i\}, 1)_{\leq} \cap (\{g, h\}, 1)_{\leq} \cap (\{a, b, d, g, f, h, i\}, 2)_{\leq} \cap (\{c, e, f\}, 2)_{\leq}.$$

We can check easily that $\mathcal{B}(M_1) \subseteq \mathcal{B}(M)$ by Lemma 1. However, by Theorem 54 below, $\mathcal{B}(M)$ is not decomposable.

Theorem 54. *The matroid base system shown in Example 53 is not decomposable.*

Proof. The matroid base system $\mathcal{B}(M)$ has one original facet-defining flat of size 4, and eight original facet-defining flats of size 3. By Theorem 29, the matroid base system $\mathcal{B}(M)$ is not 2-decomposable. By Lemma 39, an included matroid base system $\mathcal{B}(M')$ has a 3-partition in M . Since $e(3) + e(3) + e(3) = 9$ and $f(4) + 8 \times f(3) = 10$, M has no 3-partition of size $(3, 3, 3)$ by Lemma 40. Therefore the size of a 3-partition is $(2, 3, 4)$ or $(2, 2, 5)$. By Lemma 39 and Lemma 47, we can assume that one of A_1, A_2 , and A_3 has rank 1 on some included matroid base system.

Since this matroid has symmetry, we have to consider decomposability only for $ab, ac, bc, be, bg, cg, bd, bf, ce$. For each case of $X = \{b, e\}, \{c, g\}$, its graph $g(X, X^c)$ is connected. Therefore it cannot have a 3-partition with connected component X by Lemma 44. For each case of $X = \{a, b\}, \{a, c\}, \{b, c\}, \{b, g\}$, $g(X, X^c)$ has two connected components and one of them has size 1. By using Lemmas 43 and 44, we can show that there exists no included matroid base system with such a flat of rank 1.

As for $\{b, d\}$, $g(\{b, d\}, \{a, c, e, f, g, h, i\})$ has two connected components $\{f, h\}$, and $\{a, c, e, i, g\}$ since its edges are $f-h$ and $e-a-c-e-g$. However, since $\{b, d, f, h\}$ is of rank 2 on M , no included matroid base system has a 3-partition in M with a flat $\{b, d\}$. The case of $\{b, f\}$ is similar.

As for $\{c, e\}$, $g(\{c, e\}, \{a, b, d, f, g, h\})$ has three connected components $\{f\}$, $\{g, h\}$, and $\{a, b, d, i\}$. Therefore, on an included matroid M' with a flat $\{c, e\}$, by Lemma 43, $\{c, e, g, h\}$ and $\{c, e, a, b, d, i\}$ have rank 2. $g(\{g, h\}, \{a, b, d, i\})$ is connected. Since $E - \{f\}$ cannot have rank 2, $\{a, b, c, d, e, i\}$ is a flat of rank 2 on M' . Therefore, by Lemma 44, $\{a, b, d, i\}$ has rank 1 on M' . Since an included matroid M' is connected, $\{a, b, d, i\}$ is a flat. Therefore $g(\{a, b, d, i\}, \{g, h, f\})$ is connected, the rank of $\{a, b, d, i, g, h, f\}$ is 2, which contradicts the connectivity of the included matroid M' . \square

(d) Non-binary and non-2-decomposable but decomposable matroids.

The matroid in Example 46 is decomposable but not 2-decomposable.

Billera et al. [1] gave an example of a matroid decomposition consisting of three matroid base systems. However, it is also 2-decomposable.

(e) Non-binary and 2-decomposable matroids.

You can easily find a lot of examples of this type, for example, uniform matroid $U_{2,4}$.

Consider the case where $\{a, b, c, d, e, f, g, h\}$ is not a flat of the included matroid M' . Then there exists a flat of rank 2 including $\{a, b, c, d, e, f, g, h\}$. Consider the case where the included matroid $\mathcal{B}(M')$ has a flat $\{a, b, c, d, e, f, g, h, k\}$ of rank 2. We have $r'(\{g, k\}) = 1$ since $r'(\{i, k, g\}) \leq 2$ and Lemma 24 where r' is the rank function of M' . Similarly $r'(\{k, e\}) = 1$. Eventually, the included matroid $\mathcal{B}(M')$ becomes non-connected to satisfy submodularity, a contradiction. So this case cannot arise. The case where $\{a, b, c, d, e, f, g, h, i\}$ is a flat can be similarly proved, and so on. \square

Lemma 58. $\mathcal{B}(M_1)$ has no included matroid base system that has $(\{e, f, g, h, i, j, k\}, 2)_{\leq}$ as a non-original facet-defining inequality.

Proof. Assume that M_1 has an included matroid base system $\mathcal{B}(M')$ with $(\{e, f, g, h, i, j, k\}, 2)_{\leq}$ as a non-original facet-defining inequality. Since $g(\{a, b, c, d\}, \{e, f, g, h, i, j, k\})$ has two connected components $\{e, f, g, h\}$ and $\{i, j\}$, the included matroid base system $\mathcal{B}(M')$ satisfies $(\{e, f, g, h\}, 1)_{\leq}$ and $(\{i, j\}, 1)_{\leq}$ by Lemma 44. Since $g(\{e, f, g, h\}, \{a, b, c, d\})$ is connected, the included matroid base system $\mathcal{B}(M')$ satisfies $(\{a, b, c, d, e, f, g, h\}, 2)_{\leq}$ by Lemma 43. When $(\{a, b, c, d, e, f, g, h\}, 2)_{\leq}$ is a facet-defining flat, the included matroid base system $\mathcal{B}(M')$ satisfies $(\{a, b, c, d\}, 1)_{\leq}$ by Lemma 44, which contradicts the connectivity of the included matroid M' . When $(\{a, b, c, d, e, f, g, h\}, 2)_{\leq}$ is not a facet-defining flat of $\mathcal{B}(M')$, the included matroid base system $\mathcal{B}(M')$ satisfies $(\{a, b, c, d, e, f, g, h, k\}, 2)_{\leq}$ or $(\{a, b, c, d, e, f, g, h, i, j\}, 2)_{\leq}$, which contradicts the connectivity of the included matroid M' . \square

Theorem 59. The pair of M_1 and M_2 is a counterexample to Conjecture 55.

Proof. Checking $\mathcal{B}(M_2) \subseteq \mathcal{B}(M_1)$ is straightforward by Lemma 1. Assume that Conjecture 55 holds. Then there exists a matroid base system M_3 such that $\mathcal{B}(M_2) \subseteq \mathcal{B}(M_3) \subsetneq \mathcal{B}(M_1)$ and $\mathcal{B}(M_3)$ is a facet of $\mathcal{B}(M_1)$. Since $\mathcal{B}(M_2) \subseteq (\{a, b, c, d\}, 1)_{=}$, $\mathcal{B}(M_3)$ has a facet-defining inequality $(\{a, b, c, d\}, 1)_{\leq}$ or $(\{e, f, g, h, i, j, k\}, 2)_{\leq}$. By Lemma 57 and Lemma 58, $\mathcal{B}(M_1)$ has no included matroid with $\mathcal{B}(M_2)$ as a facet. \square

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